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Estimates of Asymmetric Freud Polynomials on the Real Line*

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We find bounds for the polynomials $p_n(x)$ orthogonal with respect to asymmetric Freud weights of the form $w(x)=\exp(-Q(x))$, where Q is an even degree polynomial with positive leading coefficient, by utilizing asymptotics for the recursion coefficients a_n and b_n and an approximate differential equation satisfied by $p_n(x)$.

I. INTRODUCTION AND NOTATION

Let $Q(x)$ be a polynomial of even degree with positive leading coefficient and let $w(x)=\exp(-Q(x))$ be a weight on the real line. The orthonormal polynomials $p_n(w; x)=\gamma_n x^n+\cdots$, where $\gamma_n>0$, are defined by the relation

$$\int_{-\infty}^{+\infty} p_m(w; x)\, p_n(w; x)\, w(x)\, dx=\delta_{m,n}.$$

Every system of orthogonal polynomials $\{p_n(d\alpha; x)\}_{n=0}^{\infty}$ satisfies a three-term recurrence equation

$$xp_n(d\alpha; x)=a_{n+1}\, p_{n+1}(d\alpha; x)+b_n\, p_n(d\alpha; x)+a_n\, p_{n-1}(d\alpha; x), \quad (1.1)$$

where $a_n=a_n(d\alpha)$, $b_n=b_n(d\alpha)$. In this paper we will find bounds for p_n over the real line. Our technique is to generate an approximate differential equation and then derive the estimates using this equation and asymptotics for a_n and b_n. This method is an extension of the model developed by Bonan and Clark [5, 6] to handle Freud weights of the form $w_m(x)=\exp(-x^m)$.

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For the symmetric weights $w_m(x)$ the recurrence coefficient $b_n = 0$. For the asymmetric weights that we will investigate, b_n is non-zero and introduces a great deal of complexity to the analysis. We extend Bonan and Clark's work in two ways. First, we pass to the asymmetric case, and second, we eliminate an error term in the approximate differential equation. Preliminary versions of our results were announced in [2] without proofs. For further background see the surveys of Lubinsky [9] and Nevai [14], and also Lubinsky and Saff [11] and Van Assche [20]; for specific examples of these methods applied to the Hermite polynomials and the symmetric Freud weights $w_m(x)$, see [5, 6]. See also [16, 19] for general reference.

When the meaning is clear, we write $p_n = p_n(d\alpha; x) = \gamma_n(d\alpha)x^n + \cdots$, also, we use c, c_0, c_1, etc., to denote positive constants independent of n and x, not necessarily keeping the same value from line to line. The notation $f \sim g$ will indicate that there exist positive constants c_1 and c_2 such that

$$c_1\, g(x) \leqslant |f(x)| \leqslant c_2\, g(x)$$

for the appropriate range of x.

II. The Main Results

First, the approximate differential equation.

THEOREM I. *Let $p_n(w; x)$ be the orthogonal polynomials associated with $w(x) = \exp(-Q(x))$ where $Q(x) = \sum_{k=1}^{m} d_k x^k/k$, m is an even integer, and $d_m > 0$, and let $a_n = a_n(w)$ be the recurrence coefficient in* (1.1). *Then the function*

$$z(x) := p_n(w; x)\sqrt{w(x)/A_n(x)} \tag{2.1}$$

is a solution of

$$z''(x) + \phi_n(x)\, z(x) = 0, \tag{2.2}$$

where

$$\phi_n(x) := A_n^2(x)\left(1 - \left(\frac{x - c_b}{2c_a n^{1/m}}\right)^2\right) + g_n(x), \tag{2.3}$$

$$A_n(x) := a_n \int_{\Re} p_n^2(w; t)\,\frac{Q'(t) - Q'(x)}{t - x}\, w(t)\, dt; \tag{2.4}$$

and

$$c_a := \left[\frac{d_m}{m}\binom{m-1}{m/2-1}\right]^{-1/m} \quad \textit{and} \quad c_b := -\frac{d_{m-1}}{(m-1)d_m}. \tag{2.5}$$

The error function g_n has the bounds

$$|g_n(x)| \leqslant \frac{c}{n}\left(1+\left(\frac{x-c_b}{2c_a n^{1/m}}\right)^2\right) A_n^2(x). \tag{2.6}$$

Remark. The constants c_a and c_b of (2.5), given by the Freud conjecture, are the limits of $a_n/n^{1/m}$ and b_n as n approaches infinity. They are from the expansions given in [3, Theorem 5]. See also [12].

With the approximate differential equation in hand we can find bounds for $p_n(x)$.

THEOREM II. *Let $p_n(w; x)$ be the orthonormal polynomials associated with $w(x)$ as above. Then there exists a positive constant C such that for $|x-c_b| < 2c_a n^{1/m}$ and for $n = 1, 2, ...$,*

$$p_n^2(w; x)\, w(x) \leqslant \frac{C}{\sqrt{(2c_a n^{1/m})^2 - (x-c_b)^2}}, \tag{2.7}$$

where c_a and c_b are the constants of (2.5).

We can now state the upper and lower bounds for $p_n(x)$ on the real line.

THEOREM III. *Let $p_n(w; x)$ be as above. Then*

$$\max_{x \in \Re} p_n^2(w; x)\, w(x) \sim n^{1/3 - 1/m}. \tag{2.8}$$

III. THE GENERAL DIFFERENTIAL EQUATION

Shohat [18] found that the orthogonal polynomials associated with exponential weights having the form $w(x) = (1/A(x)) \exp(\int (B(x)/A(x))\, dx)$, for fixed polynomials $A > 0$ and B, satisfy a second order differential equation. Asymptotic expressions for the recurrence coefficients $a_n(w)$ and $b_n(w)$ have allowed analysis of these differential equations to produce estimates for $p_n(w; x)$.

In 1981 Bonan observed that the Freud polynomials form a generalized Appell sequence (see [7]); it is from this observation and the recurrence that we can easily generate a differential equation for $p_n(w; x)$. This method of obtaining the differential equation is essentially due to Shohat.

LEMMA 3.1. *Let $\{p_n(w; x)\}$ be a system of orthogonal polynomials that satisfies*

$$p_n'(x) = A_n(x)\, p_{n-1}(x) - B_n(x)\, p_n(x) \tag{3.1}$$

for certain polynomials $A_n(x)$ and $B_n(x)$ of degree fixed and independent of n. Then $p_n(x)$ is a solution of

$$p_n''(x) + M(x)\, p_n'(x) + N(x)\, p_n(x) = 0, \tag{3.2}$$

where

$$M(x) := B_{n-1}(x) + B_n(x) - \frac{x - b_{n-1}}{a_{n-1}} A_{n-1}(x) - \frac{A_n'(x)}{A_n(x)} \tag{3.3}$$

and

$$N(x) := A_{n-1}(x) \left(\frac{a_n}{a_{n-1}} A_n(x) - \frac{x - b_{n-1}}{a_{n-1}} B_n(x) \right)$$

$$+ B_{n-1}(x)\, B_n(x) - \frac{A_n'(x)}{A_n(x)} B_n(x) + B_n'(x). \tag{3.4}$$

Proof. We differentiate (3.1) to obtain $p_n'' = A_n' p_{n-1} + A_n p_{n-1}' - B_n' p_n - B_n p_n'$, then apply (3.1) to replace $p_{n-1}'(x)$. Now use the recurrence (1.1) to replace the $p_{n-2}(x)$ terms and (3.1) to eliminate $p_{n-1}(x)$ terms to obtain the result. ∎

Only for small degrees have A_n and B_n been calculated explicitly (cf. [14, pp. 130–136]). For example, for $m = 2$ (see, e.g., [8, p. 49] or [19, Sect. 5.5]), $Q(x) = x^2$ gives

$$A_n(x) = 2a_n = \sqrt{2n}$$

and

$$B_n(x) = 0.$$

When $m = 4$ [1, 14], then $Q(x) = x^4/4 + q_3(x^3/3) + q_2(x^2/2) + q_1 x$ and we have

$$A_n(x) = a_n(x^2 + (q_3 + b_n)x + (a_n^2 + a_{n+1}^2 + b_n^2 + b_n q_3 + q_2))$$

and

$$B_n(x) = a_n^2(x + (b_{n-1} + b_n + q_3)).$$

For $m=6$ [17], when $Q(x)=x^6/6$ we have

$$A_n(x)=a_n(x^4+(a_n^2+a_{n+1}^2)x^2+(a_{n+1}^2(a_{n+2}^2+a_{n+1}^2+a_n^2)$$
$$+a_n^2(a_{n+1}^2+a_n^2+a_{n-1}^2)))$$

and

$$B_n(x)=a_n^2(x^3+(a_{n+1}^2+a_n^2+a_{n-1}^2)x).$$

Transforming the differential equation of Lemma 3.1 obtains

THEOREM 3.2. *Let $p_n(w;x)$ satisfy* (3.1) *and set*

$$z(x):=p_n(w;x)\sqrt{w_n(x)/A_n(x)} \tag{3.5}$$

with

$$w_n(x):=\exp\left(\int (M+A_n'/A_n)\,dx\right).$$

Then

$$z''(x)+\phi_n(x)\,z(x)=0, \tag{3.6}$$

where M is defined by (3.3), *N by* (3.4), *and*

$$\phi_n(x):=N(x)-\tfrac{1}{2}M'(x)-\tfrac{1}{4}M^2(x). \tag{3.7}$$

Proof. We apply the standard transformation to eliminate the first order term from (3.2) to generate (3.6) (see, e.g., [19, Sect. 1.8]). ∎

IV. AN APPROXIMATE DIFFERENTIAL EQUATION FOR $p_n(w;x)$

In 1984 Nevai [15] used estimates for a_n and b_n to asymptotically solve the differential equation for the polynomials orthogonal with respect to $w(x)=\exp(-x^4)$; subsequently, Sheen [17] and Bauldry [1] handled the cases $w(x)=\exp(-x^6)$ and $w(x)=\exp(-x^4+\pi_3(x))$, π_3 an arbitrary cubic polynomial, respectively. The expansions for a_n and b_n are found in

THEOREM 4.1 (Bauldry, Máté, and Nevai [3, p. 223]). *Let $a_n(w)$ and $b_n(w)$ be the recursion coefficients of* (1.1) *for $w(x)=\exp(-Q(x))$ where $Q(x)=\sum_{k=1}^m d_k x^k/k$, m is an even integer, and $d_m>0$. Then there exist constants $\eta_{i,k}$, $i=1,2$, and $k=1,2,\ldots$ such that for any $N>0$*

$$a_n(w)n^{-1/m}=c_a+\sum_{k=1}^{N}\eta_{1,k}n^{-2k/m}+o(n^{-2N/m}) \tag{4.1}$$

and

$$b_n(w)=c_b+\sum_{k=1}^{N}\eta_{2,k}n^{-2k/m}+o(n^{-2N/m}), \tag{4.2}$$

where c_a and c_b are the constants of (2.5).

Bonan and Clark [6] developed an approximate differential equation using the expansions for a_n to successfully analyze the even weight $w(x)=\exp(-x^{2m})$; here we extend their technique to the asymmetric case.

We begin with the Fourier expansion of $p_n'(x)$.

LEMMA 4.2 (cf. [6, 13]). *The polynomials $p_n(w; x)$ satisfy the relation*

$$p_n'(w;x)=A_n(x)\,p_{n-1}(w;x)-B_n(x)\,p_n(w;x), \tag{4.3}$$

with

$$A_n(x):=a_n\int_{\mathfrak{R}}p_n^2(w;t)\,\frac{Q'(t)-Q'(x)}{t-x}\,w(t)\,dt \tag{4.4}$$

and

$$B_n(x):=a_n\int_{\mathfrak{R}}p_{n-1}(w;t)\,p_n(w;t)\,\frac{Q'(t)-Q'(x)}{t-x}\,w(t)\,dt. \tag{4.5}$$

Proof. Write p_n' in a Fourier expansion in terms of the kernel $K_n(t,x)=\sum_{k=0}^{n-1}p_k(t)\,p_k(x)$ as

$$p_n'(w;x)=\int_{\mathfrak{R}}K_n(t,x)\,p_n'(w;t)\,w(t)\,dt.$$

Now integrate by parts and use the orthogonality of $p_n(t)$ to $K_n(t,x)$ to obtain

$$p_n'(w;x)=\int_{\mathfrak{R}}K_n(t,x)\,p_n(w;t)(Q'(t)-Q'(x))\,w(t)\,dt.$$

Last, we apply the Christoffel–Darboux identity (cf. [8, p. 24]) to finish the derivation. ∎

This completes the preparations for the

Proof (*Theorem* I). The proof is in three parts; first we generate the differential equation, then it is shown that $w_n(x)=w(x)$, and last, we develop the bounds for $g_n(x)$.

I. The differential equation follows directly from Theorem 3.2 and the previous lemma with the definitions

$$w_n(x) := \exp(-Q(x) + h_n(x)),$$

$$h_n(x) := Q(x) + \int \left(M(x) + \frac{A_n'(x)}{A_n(x)} \right) dx,$$

and

$$g_n(x) := N(x) - \frac{1}{2} M'(x) - \frac{1}{4} M^2(x) - A_n^2(x) \left(1 - \left(\frac{x - c_b}{2c_a n^{1/m}} \right)^2 \right),$$

where M and N are the functions defined by (3.3) and (3.4).

II (cf. [13]). The error function $h_n(x)$ is eliminated by considering the expression $M + A_n'/A_n$ from the definition above; i.e.,

$$M(x) + A_n'(x)/A_n(x) = B_{n-1}(x) + B_n(x) - \frac{x - b_{n-1}}{a_{n-1}} A_{n-1}(x).$$

Using the definitions (4.4) and (4.5), we arrive at

$$M(x) + A_n'(x)/A_n(x)$$

$$= \int_{\mathfrak{R}} (a_{n-1} p_{n-1}(t) p_{n-2}(t) + a_n p_n(t) p_{n-1}(t) - (x - b_{n-1}) p_{n-1}^2(t))$$

$$\times \frac{Q'(t) - Q'(x)}{t - x} w(t)\, dt.$$

The three-term recurrence (1.1) gives

$$= \int_{\mathfrak{R}} p_{n-1}(t)((t-x) p_{n-1}(t)) \frac{Q'(t) - Q'(x)}{t - x} w(t)\, dt$$

which simplifies to

$$M(x) + A_n'(x)/A_n(x) = \int_{\mathfrak{R}} p_{n-1}^2(t) Q'(t) w(t)\, dt - Q'(x) \int_{\mathfrak{R}} p_{n-1}^2(t) w(t)\, dt$$

$$= -Q'(x).$$

Thus, $h_n(x) \equiv 0$.

Remark. We are grateful to H. N. Mhaskar for suggesting the above technique which removes the restriction that Q be a polynomial that appeared in an earlier version of this manuscript.

III. The last step is to find bounds for $g_n(x)$; we do this by substituting the expressions for M and N into the definition of g_n to find

$$g_n(x)=A_n\left(\frac{a_n}{a_{n-1}}A_{n-1}-A_n\right)+\frac{1}{4}\left(\left(\frac{x-c_b}{c_a n^{1/m}}\right)^2 A_n^2-\left(\frac{x-b_n-1}{a_{n-1}}\right)^2 A_{n-1}^2\right)$$
$$+\frac{1}{2}\frac{A_{n-1}}{a_{n-1}}+\frac{1}{2}\left(\frac{x-b_{n-1}}{a_{n-1}}A_{n-1}\left(B_{n-1}+B_n-\frac{A'_n}{A_n}+\frac{A'_{n-1}}{A_{n-1}}\right)\right)$$
$$+\frac{1}{2}(B'_n-B'_{n-1})-\frac{1}{4}(B_{n-1}+B_n)^2$$
$$+\frac{1}{2}(3B_n+B_{n-1})\frac{A'_n}{A_n}-\frac{3}{4}\left(\frac{A'_n}{{}_n}\right)+\frac{1}{2}\frac{A''_n}{A_n}.$$

Since Q is a polynomial, we may find A_n and B_n explicitly; i.e.,

$$A_n(x)=a_n d_m x^{m-2}+a_n(b_n d_m+d_{m-1})x^{m-3}+\cdots, \tag{4.6}$$

and

$$B_n(x)=a_n^2 d_m x^{m-3}+a_n^2((b_{n-1}+b_n)\,d_m+d_{m-1})x^{m-4}+\cdots. \tag{4.7}$$

So that, after using asymptotics for a_n and b_n from (4.1) and (4.2), we have

$$|A_n-A_{n-1}|\leqslant\frac{c}{n}|A_n| \qquad \text{and} \qquad |B_n-B_{n-1}|\leqslant\frac{c}{n}|B_n|.$$

Also note that because $2p_{n-1}(t)\,p_n(t)\leqslant p_{n-1}^2(t)+p_n^2(t)$, then

$$|B_n|\leqslant\left(1+\frac{c}{2n}\right)|A_n|.$$

Substituting asymptotics for a_n and b_n and using the expressions above in g_n, we arrive at the desired estimate. ▮

V. Bounds for $p_n(w; x)$

The next step in our analysis is to generate the bounds for the polynomials p_n by using the approximate differential equation. First we need a technical lemma; it's stated in a form for arbitrary systems of orthogonal polynomials.

Lemma 5.1 [8, Sect. 1.04, p. 23]. *Let $a_n=a_n(d\alpha)$ be the recursion coefficient of* (1.1), *let $x_{n,n}<\cdots<x_{k,n}<\cdots<x_{1,n}$ be the zeros of $p_n(d\alpha; x)$ and*

let $\lambda_{k,n} = \lambda_{k,n}(d\alpha) > 0$ be the Cotes numbers of the Gaussian Quadrature for $d\alpha$. Then for $n = 1, 2, ...,$ and $k = 1, 2, ..., n$,

$$a_n^{-1} = p_n'(x_{k,n})\, p_{n-1}(x_{k,n})\, \lambda_{k,n}. \tag{5.1}$$

Remark. Combining (4.3) evaluated at $x_{k,n}$, $k = 1, 2, ..., n$, with (5.1) shows that

$$A_n(x_{k,n}) = \frac{p_n'(x_{k,n})}{p_{n-1}(x_{k,n})} = a_n\, p_n'^2(x_{k,n})\, \lambda_{k,n}, \tag{5.2}$$

which in turn gives

$$A_n(x_{k,n}) > 0. \tag{5.3}$$

This brings us to the main results of our paper.

Proof (*Theorem* II). The proof is in three parts following the model of Bonan and Clark [6]. First we use Sturm's Comparison Theorem to develop an inequality for $|x - x_{k,n}|$. The second step is to give an estimate for $z^2(x)$ by a concavity argument. The last stage is to apply the definition of $z(x)$ to obtain the result.

Remark. In order to simplify the exposition, we will assume $c_b = 0$; this is equivalent to a translation to eliminate the x^{m-1} term from Q.

I. Let $|x| < 2c_a n^{1/m}$. Consider the differential equation $z_t''(x) + \phi_n^*(t)\, z(x) = 0$ where $\phi_n^*(t) := c_n A_n^2(t)(1 - t^2/4c_a^2 n^{2/m})$ for $|x| < |t| < 2c_a n^{1/m}$; and let $x_k = k\pi/\sqrt{\phi_n^*(t)}$, $k = 0, \pm 1, \pm 2, ...,$ be the zeros of z_t. From Sturm's Comparison Theorem and the definitions of ϕ_n, ϕ_n^*, and g_n, we have the zeros of z and z_t interlace. Hence, for some zero $x_{k,n}$ of p_n we have

$$|x - x_{k,n}| \leqslant \frac{\pi}{\sqrt{\phi_n^*(t)}}.$$

Since t is arbitrary, we have

$$|x - x_{k,n}| \leqslant \frac{\pi}{\sqrt{\phi_n^*(x)}}. \tag{5.4}$$

II. Since $\phi_n > 0$, Eq. (2.2) implies that $|z|$ is concave between each pair of consecutive zeros. Hence (cf. [6]),

$$\frac{1}{2}\, |z(x)(x - x_{k,n})| \leqslant \int_{x_{k,n}}^{x} |z(t)|\, dt.$$

Applying the Cauchy–Schwarz inequality

$$\left\{\frac{1}{2} z(x)(x-x_{k,n})\right\}^2 \leqslant \int_{x_{k,n}}^{x} \frac{p_n^2(t)\, w(t)}{(t-x_{k,n})^2 A_n(x)} \int_{x_{k,n}}^{x} (t-x_{k,n})^2\, dt$$
$$\leqslant \frac{c}{A_n(x_{k,n})} \int_{x_{k,n}}^{x} \frac{p_n^2(t)\, w(t)}{(t-x_{k,n})^2}\, dt \int_{x_{k,n}}^{x} (t-x_{k,n})^2\, dt.$$

We extend the first integral to $(-\infty, +\infty)$ and evaluate both to see

$$\left\{\frac{1}{2} z(x)(x-x_{k,n})\right\}^2 \leqslant \frac{c}{A_n(x_{k,n})} \lambda_{k,n}\, p_n'^2(x_{k,n})(x-x_{k,n})^3. \tag{5.5}$$

Equations (5.2) and (5.4) now give

$$z^2(x) \leqslant \frac{c}{\sqrt{a_n^2 \phi_n^*(x)}}. \tag{5.6}$$

III. Our last step is to apply the definition of $z(x)$ to the inequality above to see that

$$p_n^2(x)\, w(x) \leqslant \frac{c}{\sqrt{a_n^2 \phi_n^*(x)/A_n^2(x)}}.$$

The result follows when the radicand is simplified to

$$p_n^2(x)\, w(x) \leqslant \frac{c}{\sqrt{4a_n^2 - x^2}}$$

and asymptotic expressions are substituted. ∎

This result is closely related to Theorem A(ii) of [10]; however, the method of the proof is significantly different. We are now ready to proceed to the proof of Theorem III. The idea of the proof is essentially the same as that used by Bonan and Clark [6]; however, the asymmetry complicates the calculations significantly. Use of new asymptotics for x_{1n} from [4] simplifies finding the upper estimate.

Proof (*Theorem* III). The upper bound follows from Theorem II along with the observation that z decreases for large x and an integral estimate for $z(x)$ for x near x_{1n} derived from $z(x) = \iint z''\, ds\, dt$. The lower bound is found using the identity (cf. [8, Sect. 1.3])

$$1 = \int_{\Re} \left(\frac{p_n(x)}{x-x_{1n}}\right)^2 \frac{w(x)}{\lambda_{1n}\, p_n'^2(x_{1n})}\, dx \tag{5.7}$$

and Bonan's observation that a significant portion of this integral occurs away from x_{1n} together with an estimate for the kernel $K_n(x)$ for x near x_{1n}.

I. *The Upper Bound*

When $|x| \leqslant x_{1n}$, we use Theorem II to obtain the bound. For $x_{1n} \leqslant |x| \leqslant 2c_a n^{1/m}$, we recall (5.5) with $k=1$ and (5.2) to have

$$z^2(x) \leqslant c\,\frac{\lambda_{1n} p_n'^2(x_{1n})}{A(x_{1n})}\,|x-x_{1n}|$$
$$\leqslant c a_n^{-1}\,|x-x_{1n}|.$$

Replacing x by $2c_a n^{1/m}$ and using the definition of $z(x)$ gives

$$p_n^2(x)\,w(x) \leqslant c\,\frac{A_n(2c_a n^{1/m})}{a_n}\,|2c_a n^{1/m} - x_{1n}|.$$

We use asymptotics for a_n from (4.1), for x_{1n} from [4, Theorem I], and A_n's expansion (4.6) to obtain

$$p_n^2(x)\,w(x) \leqslant c(n^{1/m})^{m-2} n^{-2/3}$$

which yields the desired bound. Since we do not know the largest zero of $\phi_n(x)$, i.e., the exact point above which $z(x)$ is always concave down, we estimate $z(x)$ for x around $2c_a n^{1/m}$ as follows. Let $|x-2c_a n^{1/m}| \leqslant cn^{-1+1/m}$. For some t_0 such that $|t-t_0| \leqslant cn^{-2/3+1/m}$ and $z'(t_0)=0$,

$$|z(x) - z(2c_a n^{1/m})| \leqslant \left(1+\frac{c}{n}\right) \int_{2c_a n^{1/m}}^{x} \int_{t_0}^{t} \phi_n^*(s)\,|z(s)|\,ds\,dt$$
$$\leqslant c \int_{2c_a n^{1/m}}^{x} \int_{t_0}^{t} A_n^2(s) \left(1 - \left(\frac{s}{2c_a n^{1/m}}\right)^2\right) |z(s)|\,ds\,dt.$$

Since $A_n(x) \sim a_n x^{m-2}$ and $1-(s/2c_a n^{1/m})^2 \sim n^{-2/3}$,

$$|z(x) - z(2c_a n^{1/m})| \leqslant cn^{2-2/m} n^{-2/3} \int_{2c_a n^{1/m}}^{x} \int_{t_0}^{t} |z(s)|\,ds\,dt$$
$$\leqslant cn^{4/3-2/m} \int_{2c_a n^{1/m}}^{x} \left(\int_{t_0}^{t} ds \int_{-\infty}^{\infty} p_n^2(s)\,w(s)\,ds \right)^{1/2} dt,$$

which yields

$$|z(x) - z(2c_a n^{1/m})| \leqslant cn^{4/3-2/m} |t-t_0|^{1/2}\,|x-2c_a n^{1/m}|$$

giving the required bound. For large $|x|$, we observe that $z(x)$ is decreasing to finish the derivation.

II. *The Lower Bound*

Set $\Delta = \{x\colon |x - x_{1n}| \leqslant \varepsilon n^{1/3 - 1/m}\}$ and let $K_n(x) = K_n(x, x)$. We use the inequality

$$p^2(x) \leqslant K_n(x) \int_{-\infty}^{\infty} p^2(t)\, w(t)\, dt$$

for $p(x)$ a polynomial of degree less than n (cf. [18, Sect. 3.1]) with $p(x) = p_n(x)/(x - x_{1n})$ to obtain

$$\int_\Delta \left(\frac{p_n(x)}{x - x_{1n}}\right)^2 \frac{w(x)}{\lambda_{1n}\, p_n'^2(x_{1n})}\, dx \leqslant \int_\Delta K_n(t)\, w(t)\, dt$$

which is

$$\leqslant 2\varepsilon n^{1/3 - 1/m} \max_{x \in \Delta} (K_n(x)\, w(x)).$$

The same procedure used in [6] gives the estimate needed for $K_n(x)$ over Δ,

$$\max_{x \in \Delta} (K_n(x)\, w(x)) \leqslant c n^{-(1/3 - 1/m)}.$$

Then, with 1/4c, we have

$$\int_\Delta \left(\frac{p_n(x)}{x - x_{1n}}\right)^2 \frac{w(x)}{\lambda_{1n}\, p_n'^2(x_{1n})}\, dx \leqslant \frac{1}{2},$$

which yields

$$\frac{1}{2} \leqslant \int_{\mathfrak{R}/\Delta} \left(\frac{p_n(x)}{x - x_{1n}}\right)^2 \frac{w(x)}{\lambda_{1n}\, p_n'^2(x_{1n})}\, dx.$$

Hence

$$\frac{1}{2} \leqslant c(\varepsilon n^{1/3 - 1/m})^{-1} \max_{x \in \Delta} (p_n^2(x)\, w(x)),$$

from which the result follows. ∎

In a sequel we intend to sharpen the analysis of the error function g_n of the differential equation. It appears to be possible to develop asymptotics for $p_n(w; x)$ from the differential equation given improvements to this error term.

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